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The non-compact arithmetic generalised triangle groups

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Abstract

All non-compact arithmetic Kleinian groups with two elliptic generators are determined. The non-compact arithmetic generalised triangle groups are necessarily included and 15 of the 21 groups obtained turn out to be generalised triangle groups, the remainder being quotients of generalised triangle groups. The generalised triangle groups all yield orbifolds with a simple singular set in S^3.

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1. Introduction

A generalised triangle group is a two-generator group with a presentation of the form

$$\langle a, b \mid a^p = b^q = R(a,b)^r = 1 \rangle,$$

where $R(a,b)$ is a cyclically reduced word in the free product on a, b which involves both a and b and p, q, r are all integers which are at least 2. The classical triangle groups occur for $R(a,b) = ab$ and they are discrete subgroups of the isometry groups of $\mathbb{S}^2, \mathbb{R}^2$ or $\mathbb{H}^2$ according as the sum $1/p + 1/q + 1/r$ is greater than 1, equal to 1 or less than 1, respectively. Various generalised triangle groups have been shown to have faithful representations as discrete subgroups of $PSL(2,\mathbb{C})$, the group of orientation-preserving isometries of $\mathbb{H}^3$ [1,9–12].

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In this paper those generalised triangle groups Γ which are arithmetic subgroups of $PSL(2,\mathbb{C})$ such that $\mathbb{H}^3/\Gamma$ is non-compact are obtained. In fact, we determine all non-cocompact arithmetic Kleinian groups which are generated by two elliptic elements. This is part of a wider programme to determine all two-generator arithmetic Kleinian groups. When the generators are either elliptic or parabolic, it has been shown that there are finitely many such groups up to conjugacy and the groups themselves have been determined when at least one generator is parabolic [3,5,14]. When both generators are parabolic, there are four such groups and they are all two bridge knot or link groups [5] and when one generator is parabolic and the other elliptic, there are 16 such groups [3]. In these cases, the quotient spaces are also non-compact and the groups are commensurable with Bianchi groups. In the cases considered in this paper of arithmetic Kleinian groups with two elliptic generators and non-compact quotient space, there are 21 such presentations, and surprisingly all groups are commensurable with one of the two Bianchi groups $PSL(2,O_1)$ and $PSL(2,O_3)$. The groups are listed in Table 1 with a presentation on two elliptic generators from which those that are obviously generalised triangle groups may be identified. The structure of the corresponding orbifolds is discussed in Section 7.

Note that for each $m \geqslant 1$ the generalised triangle groups with presentation $\langle a,b \mid a^3 = b^3 = ((ab)^m(a^{-1}b^{-1})^m)^3 = 1\rangle$ all have a discrete and faithful representation in $PSL(2,\mathbb{C})$ which is non-compact and of finite covolume. Thus, there are infinitely many non-compact generalised triangle groups and some hypothesis, such as arithmeticity, is necessary to obtain finiteness theorems. In fact, this particular family of groups is arithmetic only when $m = 1$ and the underlying orbifold is given in Fig. 3 with $p = q = r = 3$. Related examples are discussed in Section 7.

2. Kleinian groups and arithmeticity

A Kleinian group Γ is a discrete subgroup of $PSL(2,\mathbb{C})$, the group of all orientation-preserving isometries of $\mathbb{H}^3$. All groups considered will be non-elementary, which means that they are not virtually abelian. The quotient space $\mathbb{H}^3/\Gamma$ is a hyperbolic orbifold or, if Γ is torsion-free, a hyperbolic manifold. The linear fractional action of a matrix

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$$

on $\hat{\mathbb{C}}$ extends via the Poincaré extension to give an isometry f of the upper-half-space model of $\mathbb{H}^3$ where $\partial\mathbb{H}^3 = \hat{\mathbb{C}}$. For isometries f, g the complex numbers

$$\beta(f) = \mathrm{tr}^2(f) - 4, \quad \beta(g) = \mathrm{tr}^2(g) - 4, \quad \gamma(f,g) = \mathrm{tr}[f,g] - 2$$

are called the *parameters* of the two generator group $\langle f,g\rangle$ and written

$$par(\langle f,g\rangle) = (\gamma(f,g), \beta(f), \beta(g)). \tag{1}$$

Note that these parameters are independent of the choice of matrix representatives for f, g in $SL(2,\mathbb{C})$ and they determine $\langle f,g\rangle$ uniquely up to conjugacy whenever $\gamma(f,g) \neq 0$ [7].

Table 1
All non-compact arithmetic groups with two elliptic generators

Γ_i	p	q	γ	d	$\langle x^p = y^q = \cdots = 1\rangle$.	Notation	Covolume
1	2	3	$(-3+\sqrt{3}\mathrm{i})/2$	3	$((xy^{-1})^2(x^{-1}y)^2)^3$	$\Gamma_{10}\rtimes\mathbb{Z}_2$	0.33831
2	2	3	$(-1+\sqrt{3}\mathrm{i})/2$	3	$((xy^{-1})^3(x^{-1}y)^3)^3$	$\Gamma_{11}\rtimes\mathbb{Z}_2$	0.67664
3	2	3	$(-5+\sqrt{3}\mathrm{i})/2$	3	$(xyx^{-1}yxy^{-1})^6$	$\tilde{\Gamma}(2,3;6)$	0.67664
4	2	4	$-1+\mathrm{i}$	1	$((xy^{-1})^2(x^{-1}y)^2)^2$	$\Gamma_{17}\rtimes\mathbb{Z}_2$	0.45798
5	2	4	$-2+\mathrm{i}$	1	$(xyx^{-1}yxy^{-1})^4$	$\tilde{\Gamma}(2,4;4)$	0.91596
6	2	4	i	1	$((xy^{-1})^3(x^{-1}y)^3)^2$	$\Gamma_{18}\rtimes\mathbb{Z}_2$	0.91596
7	2	6	$(-1+\sqrt{3}\mathrm{i})/2$	3	$((yx)^2(y^{-1}x)^2)^3$ $=((yx)^2(y^{-1}x)^2y)^2$ $=(xy^{-1}x(yx)^2(y^{-1}x)^2)^2$ $=(xy^{-1}x(yx)^2(y^{-1}x)^2y)^2$	$\Gamma_{20}\rtimes\mathbb{Z}_2$	0.25374
8	2	6	$(1+\sqrt{3}\mathrm{i})/2$	3	$([y,x](yx)^2[y^{-1},x]y)^2$ $=([y,x](yx)^2[y^{-1},x])^2\ \cdot$ $xy^{-2}[x,y][x,y^{-1}](xy^{-1})^2x$	$\Gamma_{21}\rtimes\mathbb{Z}_2$	0.50748
9	2	6	$(-3+\sqrt{3}\mathrm{i})/2$	3	$(xyx^{-1}yxy^{-1})^3$	$\tilde{\Gamma}(2,6;3)$	1.01496
10	3	3	-3	3	$[x,y]^3$	$\Gamma(3,3;3)$	0.67664
11	3	3	$-2+\sqrt{3}\mathrm{i}$	3	$(xyx^{-1}yxy^{-1})^3$	$\tilde{\Gamma}(3,3;3)$	1.35328
12	3	4	-2	1	$[x,y]^2$	$\Gamma(3,4;2)$	0.30532
13	3	4	$2\mathrm{i}$	1	$((xy^{-1})^2(x^{-1}y)^2)^2$		1.5266
14	3	6	-1	3	$[x,y]^3=([x,y]x)^2$ $=(y^{-1}[x,y])^2$ $=(y^{-1}[x,y]x)^2$	$3-3-6$	0.08458
15	3	6	$-1+\sqrt{3}\mathrm{i}$	3	$(xyx^{-1}yxy^{-1})^2$	$\tilde{\Gamma}(3,6;2)$	1.6916
16	3	6	$2+\sqrt{3}\mathrm{i}$	3	$((xy)^2xy^{-1}(x^{-1}y^{-1})^2)^2$		2.70656
17	4	4	-2	1	$[x,y]^4$	$\Gamma(4,4;2)$	0.91596
18	4	4	$-1+2\mathrm{i}$	1	$(xyx^{-1}yxy^{-1})^2$	$\tilde{\Gamma}(4,4;2)$	1.83192
19	4	6	-1	3	$[x,y]^3=([x,y]x)^2$ $=(y^{-1}[x,y])^2$ $=(y^{-1}[x,y]x)^2$	$4-3-6$	0.21145
20	6	6	-1	3	$[x,y]^3=([x,y]x)^2$ $=(y^{-1}[x,y])^2$ $=(y^{-1}[x,y]x)^2$	$6-3-6$	0.50748
21	6	6	$\sqrt{3}\mathrm{i}$	3	$(y^{-1}x)^2y[x^{-1},y][x,y]\ \cdot$ $[x,y^{-1}]x^{-1}$ $=([y^{-1},x]yx^2)^2$	$\tilde{\Gamma}(2,6;3)$	1.01496

If f is parabolic, then $\beta(f)=0$; if g is elliptic, then some power of g is primitive and for a primitive elliptic g, $\beta(g) = -4\sin^2 \pi/n$ where n is the order of g. Thus if $G=\langle f,g\rangle$ is a Kleinian group where f,g are elliptic of orders p and q, respectively, then

$$par(G) = (\gamma, -4\sin^2 \pi/p, -4\sin^2 \pi/q). \tag{2}$$

For fixed p,q the space of all such discrete groups is determined by the single complex parameter $\gamma(f,g)$ and it is this value which we seek (see Table 1).

Note that when $p=2$, the subgroup $\langle g, fgf\rangle$ is of index 2 in $\langle f,g\rangle$ and has parameters

$$(\gamma(f,g)(\gamma(f,g)-\beta(g)), \beta(g), \beta(g)). \tag{3}$$

Conversely, each group generated by a pair of elements g_1, g_2 of the same order can be extended by elements of order 2 which conjugate g_1 to $g_2^{\pm 1}$ [8].

If G is a finitely generated non-elementary subgroup, then its trace field $\mathrm{tr}(G) := \mathbb{Q}(\pm \mathrm{tr}\, g\colon g\in G)$ is well defined. If $G^{(2)}$ denotes the finite index subgroup $\langle g^2\colon g\in G\rangle$, then $\mathrm{tr}(G^{(2)})$ is an invariant of the commensurability class of G, denoted by kG and called the *invariant trace field* of G [17]. In addition,

$$AG := \{\textstyle\sum a_i\gamma_i\colon \gamma_i \in G^{(2)}, a_i\in kG\}$$

is a quaternion algebra over kG and is also an invariant of the commensurability class of G. Note that, for $f,g\in G, f^2, g^2, [f,g]$ all lie in $G^{(2)}$ so that the parameters of $G=\langle f,g\rangle$, i.e. $\beta(f),\beta(g),\gamma(f,g)$ all lie in kG.

We now recall the definition of an arithmetic Kleinian group. Let k be a number field and let A be a quaternion algebra over k. If v is a place of k, let k_v denote the completion of k at v. Then A is said to be ramified at v if the quaternion algebra $A\otimes_k k_v$ is a division algebra over k_v. Now suppose that k has exactly one complex place and A is ramified at all real places. Let ρ be an embedding of A into $M_2(\mathbb{C})$, $\mathcal{O}$ an order of A and $\mathcal{O}^1$ the elements of norm 1 in $\mathcal{O}$. Then $P\rho(\mathcal{O}^1)$ is a finite covolume Kleinian group and the totality of arithmetic Kleinian groups consists of all groups commensurable with some such $P\rho(\mathcal{O}^1)$ [2,18].

The field of definition of an arithmetic Kleinian group is recovered as its invariant trace field and the quaternion algebra is the invariant quaternion algebra. One then deduces that two arithmetic Kleinian groups are commensurable up to conjugacy if their invariant quaternion algebras are isomorphic [15]. More generally, one has the following [6].

Theorem 2.1. *Let G be a finitely generated non-elementary subgroup of the group $PSL(2,\mathbb{C})$ such that*

- *kG is a number field with exactly one complex place*,
- $\mathrm{tr}(G)$ *consists of algebraic integers*,
- *AG is ramified at all real places of kG.*

Then G is a subgroup of an arithmetic Kleinian group.

If G is a non-elementary Kleinian group which contains a parabolic element g, then $g^2-1\in AG$ is non-invertible and AG cannot be a division algebra. In such cases, AG cannot be ramified at any places.

From this one obtains

Theorem 2.2. *Let G be a non-cocompact arithmetic Kleinian group. Then $kG = \mathbb{Q}(\sqrt{-d})$ is a quadratic imaginary number field, $AG \cong M_2(\mathbb{Q}(\sqrt{-d}))$ and G is commensurable, up to conjugacy, with the Bianchi group $PSL(2, O_d)$, where O_d is the ring of integers in $\mathbb{Q}(\sqrt{-d})$.*

3. Two generator groups

Now let us assume that G is generated by two elliptic elements f, g of orders p, q where $p \leqslant q$ and that G is a non-cocompact arithmetic Kleinian group. Since $\beta(f), \beta(g) \in kG$, which is quadratic imaginary, then $p, q \in \{2,3,4,6\}$. Furthermore, we will assume that $p \geqslant 3$ since the groups with $p = 2$ are commensurable with groups where $p = q \geqslant 3$. In these cases, $kG = \mathbb{Q}(\mathrm{tr} f^2 g^2)$ (e.g. [5]). In addition, we can obtain a Hilbert symbol for the invariant quaternion algebra as

$$AG \cong \left(\frac{\beta_1(\beta_1 + 4), (\beta_1 + 4)(\beta_2 + 4)\gamma}{kG} \right), \tag{4}$$

where $\gamma = \gamma(f, g)$ and $\{\beta_1, \beta_2\} = \{\beta(f), \beta(g)\}$ (e.g. [6]).

We now normalise our choice of matrix representatives for the elements f, g. Thus, $\langle f, g \rangle$ is conjugate in $PSL(2,\mathbb{C})$ to a subgroup generated by the images of

$$X = \begin{pmatrix} \cos \pi/p & \mathrm{i} \sin \pi/p \\ \mathrm{i} \sin \pi/p & \cos \pi/p \end{pmatrix}, \quad Y = \begin{pmatrix} \cos \pi/q & \mathrm{i}t \sin \pi/q \\ \mathrm{i}t^{-1} \sin \pi/q & \cos \pi/q \end{pmatrix}. \tag{5}$$

Here t is a complex parameter and by further conjugation, if necessary, we can assume that $|t| < 1$.

If G is to be arithmetic, then it has finite covolume and hence Euler characteristic zero. Thus G cannot be a free product of cyclic groups. However, if the isometric circles of g lie inside the intersection of the isometric circles for f, then the Klein combination theorem shows that $\langle f, g \rangle$ is a free product. Thus, with the normalisation above, if the inequalities

$$|\sin(\pi/q)\cos(\pi/p) \pm t\cos(\pi/q)\sin(\pi/p)| + |t|\sin(\pi/p) < \sin(\pi/q) \tag{6}$$

hold for both choices of sign, then the group $G \cong \langle f \rangle * \langle g \rangle$. These, in particular, hold if

$$|t| < \frac{\sin(\pi/p)\sin(\pi/q)}{(1 + \cos(\pi/p))(1 + \cos(\pi/q))}. \tag{7}$$

Refinements of this argument will be used later in Section 5.1.

4. Candidates

Since the elements $[f, g]$ and $f^2 g^2 \in G^{(2)}$, it follows that

$$\gamma(f, g) = \mathrm{tr}[X, Y] - 2 = (t - t^{-1})^2 \sin^2 \frac{\pi}{p} \sin^2 \frac{\pi}{q}, \tag{8}$$

$$\alpha(f, g) = \mathrm{tr}\, X^2 Y^2 = 2\cos\frac{2\pi}{p}\cos\frac{2\pi}{q} - \sin\frac{2\pi}{p}\sin\frac{2\pi}{q}(t + t^{-1}) \tag{9}$$

are algebraic integers in the quadratic imaginary field $kG = \mathbb{Q}(\sqrt{-d})$. Recall that, for each pair of integers p, q the parameter γ determines the conjugacy class of the group, so our strategy is to use the facts that γ and α are quadratic integers for which inequality (7) must fail. This restricts the values of d and gives a finite number of possibilities for α. Those that satisfy inequality (6) are then also rejected and we are left with a finite number of candidate values of γ. Not all of these will correspond to arithmetic Kleinian groups and we use three further arguments in the elimination of further values of γ. These arguments are given in Section 5, but in the tables of candidate γ values, those values of γ to be eliminated will be labelled as *free*, *arithmetic* or *special* according to which of the three arguments is used.

For the moment, we give some details in one case and then merely tabulate the results as all cases proceed in the same way. Tables 2–7 appear below.

Suppose that $p = q = 3$. In this case inequality (7) becomes $|t| < \frac{1}{3}$ and (6) becomes

$$|t \pm 1| + 2|t| < 2. \tag{10}$$

We also have

$$\alpha = \tfrac{1}{2} - \tfrac{3}{4}(t + t^{-1}) = \tfrac{1}{2}(m + n\sqrt{-d}), \tag{11}$$

where $m \equiv n \pmod 2$ and $m \equiv n \equiv 0 \pmod 2$ if $d \not\equiv 3 \pmod 4$. Since we can assume that $|t| < 1$,

$$|\alpha| \leqslant \frac{1}{2} + \frac{3}{4} + \frac{3|t^{-1}|}{4}.$$

This obviously holds for $|\alpha| \leqslant \frac{5}{4}$. If $|\alpha| > \frac{5}{4}$, then $|t| \leqslant 3/(4|\alpha| - 5)$. In that case, if α is such that

$$\frac{3}{4|\alpha| - 5} < \frac{1}{3}$$

then the corresponding group will be a free product. Thus, we can assume that $|\alpha|^2 \leqslant 49/4$, i.e. $m^2 + n^2 d \leqslant 49$. Note that $kG = \mathbb{Q}(\alpha)$ so that α cannot be real and so $n \neq 0$. Note also that replacing n by $-n$, replaces t by its conjugate and hence γ by its conjugate. But the group defined by $\bar{\gamma}$ is

Table 2
Candidates for $p = 3$ and $q = 3$

d	γ	Eliminated
1	$-1 + 3i$	Arithmetic
1	$-3 + i$	Arithmetic
2	$-4 + \sqrt{2}i$	Arithmetic
3	-3	
3	$1 + 2\sqrt{3}i$	Free
3	$-2 + \sqrt{3}i$	
7	$-3 + \sqrt{7}i$	Free

Table 3
Candidates for $p = 4$ and $q = 4$

d	γ	Eliminated
1	-2	
1	$-1 + 2i$	
1	$2 + 4i$	Free
2	-3	Special
2	$1 + 4\sqrt{2}i$	Arithmetic
2	$-2 + 2\sqrt{2}i$	Arithmetic
3	$(1 + 3\sqrt{3}i)/2$	Arithmetic
3	$(-3 + \sqrt{3}i)/2$	Arithmetic
3	$-3 + 2\sqrt{3}i$	Arithmetic
3	$(9 + 5\sqrt{3}i)/2$	Free
3	-4	Special
7	$(-1 + 3\sqrt{7}i)/2$	Arithmetic
7	$(-5 + \sqrt{7}i)/2$	Arithmetic
11	$(-3 + 3\sqrt{11}i)/2$	Arithmetic
11	$(-7 + \sqrt{11}i)/2$	Arithmetic
15	$(-9 + \sqrt{15}i)/2$	Arithmetic

isomorphic to that defined by γ [8], so we can assume that $n \geqslant 1$. This applies in all cases. In the above case, if $d \not\equiv 3 \pmod 4$, then the possible values for d are $\leqslant 10$ while, if $d \equiv 3 \pmod 4$ we have $d \leqslant 47$. For each of these values of d, the values of m and n are also restricted. Thus for $d = 10$ for example, the only possibilities are $\alpha = \sqrt{10}i, \pm 1 + \sqrt{10}i$. For each of these we can solve the quadratic equation arising from (11), i.e.

$$t^2 + \frac{4\alpha - 2}{3}t + 1 = 0$$

choosing the root t such that $|t| < 1$. We can then eliminate those α for which t satisfies (6). For each of the remaining values of α, we determine γ since

$$\gamma = \alpha^2 - \alpha - 2 \tag{12}$$

Table 4
Candidates for $p = 6$ and $q = 6$

d	γ	Eliminated
1	$-1+i$	Arithmetic
1	$1+3i$	Arithmetic
1	$5+5i$	Arithmetic
1	$-2+6i$	Free
1	$-4+2i$	Free
2	$3\sqrt{2}i$	Arithmetic
2	$-2+\sqrt{2}i$	Arithmetic
2	$4+5\sqrt{2}i$	Free
3	-1	
3	$\sqrt{3}i$	
3	$3+2\sqrt{3}i$	Free
3	$8+3\sqrt{3}i$	Free
3	$-1+3\sqrt{3}i$	Free
3	$-3+\sqrt{3}i$	Free
3	$3+5\sqrt{3}i$	Free
5	$-3+3\sqrt{5}i$	Free
7	-2	Arithmetic
7	$-1+\sqrt{7}i$	Arithmetic
7	$2+2\sqrt{7}i$	Arithmetic
7	$7+3\sqrt{7}i$	Free
11	-3	Arithmetic
11	$1+2\sqrt{11}i$	Free
11	$-2+\sqrt{11}i$	Free
15	$2\sqrt{15}i$	Free
15	$-3+\sqrt{15}i$	Free
19	$-1+2\sqrt{19}i$	Free
19	$-4+\sqrt{19}i$	Free

Table 5
Candidates for $p = 3$ and $q = 4$

d	γ	Eliminated
1	-2	
1	$2i$	
1	$-3+2i$	Free
2	$-2+\sqrt{2}i$	Arithmetic
2	$2+3\sqrt{2}i$	Arithmetic
3	-3	Special
3	$-1+2\sqrt{3}i$	Free
5	$-2+\sqrt{5}i$	Free
6	$-4+\sqrt{6}i$	Free

Table 6
Candidates for $p = 3$ and $q = 6$

d	γ	Eliminated
1	$-3+3i$	Free
3	-1	
3	$-1+\sqrt{3}i$	
3	$2+\sqrt{3}i$	
3	-3	Special
3	$3\sqrt{3}i$	Free
3	$-4+2\sqrt{3}i$	Free
6	$-2+\sqrt{6}i$	Arithmetic
15	-2	Special
15	$1+\sqrt{15}i$	Arithmetic
39	-4	Special
39	$-1+\sqrt{39}i$	Free

Table 7
Candidates for $p = 4$ and $q = 6$

d	γ	Eliminated
1	-2	Special
1	$4 + 6i$	Free
2	$-2 + 3\sqrt{2}i$	Free
3	-1	
3	$5 + 2\sqrt{3}i$	Free
3	$-1 + 2\sqrt{3}i$	Free
6	$\sqrt{6}i$	Arithmetic
6	$-3 + 2\sqrt{6}i$	Arithmetic
15	-3	Special
15	$3 + 2\sqrt{15}i$	Arithmetic

from (8) and (9). In this case, whenever α is an integer, so is γ. For other cases, we only select those values of α for which γ is also an integer.

In all cases, with various adjustments, this yields the values of p, q, γ given in the tables.

5. Further refinements

In this section, the three further arguments used to eliminate values of γ as candidates for two generator arithmetic Kleinian groups which are not cocompact are explained. We use the terminology given in the tables.

5.1. Free

This is a refinement of the main argument used in the preceding section involving inequality (6). Inequality (6) implies that a fundamental region for $\langle g \rangle$ which consists of the exterior of the isometric circles of g contains the complement of the fundamental region for $\langle f \rangle$ which is the intersection of the isometric circles for f. This arrangement of fundamental regions implies that $\langle f, g \rangle$ is isomorphic to the free product $\langle f \rangle * \langle g \rangle$. This is the Klein combination theorem, see [16, Chapter VII]. There is no assumption necessary regarding the connectness of the fundamental sets. When the isometric circles of g do not lie in the intersection of the isometric circles of f, we can attempt to find a new fundamental domain for $\langle f \rangle$ by cutting and pasting so that this new fundamental domain still has the property that its complement lies in the intersection of the fundamental circles of g. This works in most cases; however sometimes we must further cut and paste the fundamental region for g to find fundamental regions with the required combinatorial properties.

We illustrate this with the two examples from the tables when both generators have order 3. In these cases we have

$$f \approx \begin{pmatrix} 1/2 & i\sqrt{3}/2 \\ i\sqrt{3}/2 & 1/2 \end{pmatrix}, \quad g \approx \begin{pmatrix} 1/2 & i\sqrt{3}/2 \\ it^{-1}\sqrt{3}/2 & 1/2 \end{pmatrix}.$$

The isometric circles of f bound the disks $F_\pm = \{z\colon |z \pm i/\sqrt{3}| \leqslant 2/\sqrt{3}\}$ and those for g bound the disks $G_\pm = \{z\colon |z \pm it/\sqrt{3}| \leqslant 2|t|/\sqrt{3}\}$. We may assume that t (as defined at (8)) has positive real part and modulus no more than 1. The preimage of the disk G_- under f is found by reflecting G_- in the disk F_- and then in the real axis. It is the disk $G'_- = \{z\colon |z + \omega| \leqslant r\}$, where

$$\omega = \frac{i}{\sqrt{3}}\left(1 - \frac{4(\bar{t}+1)}{|t+1|^2 - 4|t|^2}\right), \quad r = \frac{8}{\sqrt{3}}\frac{|t|}{|t+1|^2 - 4|t|^2}.$$

Similarly we may compute the image G'_+ of G_+.

Example 1. In the first example we have $\gamma = -3 + i\sqrt{7}$. This gives the value $t = 0.1952\ldots - i0.3653\ldots$. Parts of the isometric circles of g protrude from $F_+ \cap F_-$ (a fundamental domain for f). These parts are equivalent under f to the sets $G'_- \cap (F_+ \cap F_-)$ and $G'_+ \cap (F_+ \cap F_-)$. This set is disjoint from the isometric circles of g for this particular value of t (see Fig. 1).

A fundamental domain for f is easily seen to be

$$\Delta = ((F_+ \cap F_-) \cup (G_+ \cup G_-)) \backslash (G'_+ \cup G'_-)$$

which contains the isometric circles of g by construction. It follows that $\langle f, g \rangle$ is free on generators.

Example 2. In this case we have $\gamma = 1 + i2\sqrt{3}$ giving $t = 0.3216\ldots - i0.1835\ldots$. Here we are not so lucky that the sets $G'_\pm \cap (F_+ \cap F_-)$ are disjoint from the isometric circles of g. We proceed to cut a bit off the fundamental domain of g (the exterior of the isometric circles) and attach an equivalent piece (under $\langle g \rangle$) disjoint from $G'_\pm$ and inside $F_+ \cap F_-$. The pieces we add are $U = G'_- \cap G_+ \cap \{\Re(z) \geqslant 0\}$ and $V = G'_+ \cap G_- \cap \{\Re(z) \leqslant 0\}$. Note that f is normalised so as to preserve the imaginary axis. We remove equivalent sets to these pieces $g^{-1}(U) = U'$ and $g(V) = V'$ which lie inside $F_+ \cap F_-$ and are disjoint from $G'_\pm$ (see Fig. 2).

Then the two fundamental sets with the correct combinatorial properties are

$$\Delta(f) = (F_+ \cap F_-) \cup (G_- \cap \{\Re(z) \geqslant 0\}) \cup (G_+ \cap \{\Re(z) \leqslant 0\})$$
$$\backslash (G'_- \cap \{\Re(z) \geqslant 0\}) \cup (G'_+ \cap \{\Re(z) \leqslant 0\}),$$
$$\Delta(g) = ((\bar{\mathbb{C}} \backslash G_\pm) \cup (U \cup V)) \backslash (U' \cup V').$$

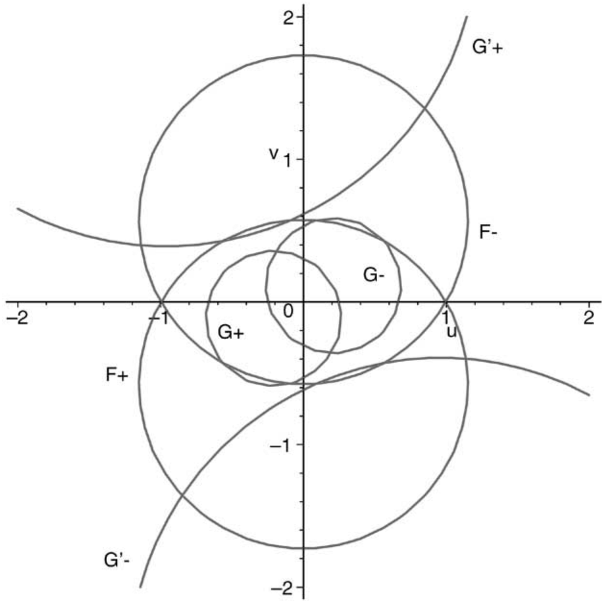

Fig. 1.

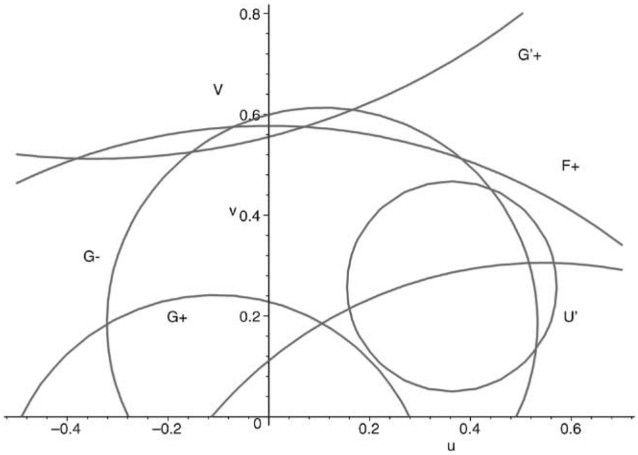

Fig. 2.

There are a couple of remarks worth making. Firstly, isometric circles are not conjugacy invariants. Choosing different conjugacies of the group $\langle f, g\rangle$ (which amounts to choosing different matrix normalisations) can significantly simplify the combinatorial patterns of the intersecting isometric circles. Even interchanging p and q (when they are different) sometimes leads to great simplification. Secondly, together with J. McKenzie, we have implemented a modified version of the Dirichlet domain procedure in J. Week's program Snappea to check that all the groups we eliminate in the manner above actually have a Dirichlet domain which meets the sphere at infinity in an open set. Such groups act discontinuously on this sphere and cannot be of finite covolume and in particular they cannot be arithmetic.

5.2. Arithmetic

From Theorem 2.2, the invariant quaternion algebra AG must be isomorphic to $M_2(\mathbb{Q}(\sqrt{-d}))$ when $kG = k = \mathbb{Q}(\sqrt{-d})$. From the expression for the Hilbert symbol for AG given at (4), it follows that $AG \cong M_2(k)$ if and only if the quadratic form

$$q(x,y) = \beta_1(\beta_1 + 4)x^2 + (\beta_1 + 4)(\beta_2 + 4)\gamma y^2 \tag{13}$$

represents 1 in k [13, p. 58].

By the Hasse–Minkowski theorem, q represents 1 in k if and only if q represents 1 in $k_{\mathscr{P}}$, for all $\mathscr{P}$, where $k_{\mathscr{P}}$ is the $\mathscr{P}$-adic field formed by completing k at the valuation corresponding to the prime ideal $\mathscr{P}$ [13, p. 168]. Let $N(\mathscr{P}) = |R_k/\mathscr{P}|$. Then $\mathscr{P}$ is called dyadic or non-dyadic according as $N(\mathscr{P})$ is even or odd.

Lemma 5.1. *Lep $\mathscr{P}$ be a non-dyadic prime ideal in R_k, and let $q(x,y) = ax^2 + by^2$ with $a, b \in R_k$. Then*
- *if $a, b \notin \mathscr{P}$, then q represents 1 in $k_{\mathscr{P}}$,*
- *if $a \notin \mathscr{P}, b \in \mathscr{P}\backslash\mathscr{P}^2$, then q represents 1 in $k_{\mathscr{P}}$ if and only if a is a square* mod $\mathscr{P}$,
- *if $a, b \in \mathscr{P}\backslash\mathscr{P}^2$, then q represents 1 in $k_{\mathscr{P}}$ if and only if $-a^{-1}b$ is a square* mod $\mathscr{P}$ [13, p. 147].

The dyadic cases are more complicated (see Example 2 below). To eliminate the values of γ labelled *arithmetic* in the tables, we show that, for these values $AG \ncong M_2(k)$. To do this, we need to find a prime ideal $\mathscr{P}$ such that q does not represent 1 in $k_{\mathscr{P}}$, making use of the above lemma. Most cases can be eliminated using non-dyadic primes as in Example 1 below, but in some cases, like Example 2 below, we need to use dyadic primes.

Example 1. Let $p = q = 3$ so that $\beta_1 = \beta_2 = -1$. Consider the case where $d = 1$ and $\gamma = -3 + \mathrm{i}$. Then, removing squares, q has the form

$$q(x,y) = -3x^2 + (-3 + \mathrm{i})y^2. \tag{14}$$

Now $-3 + \mathrm{i} = (1 + \mathrm{i})(-1 + 2\mathrm{i})$. Choose $\mathscr{P} = (-1 + 2\mathrm{i})O_1$ so $N(\mathscr{P}) = 5$. But -3 is not a square mod $\mathscr{P}$ so that q does not represent 1. Thus AG is not isomorphic to $M_2(\mathbb{Q}(i))$.

Example 2. Let $p = q = 4$ so that $\beta_1 = \beta_2 = -2$. Consider the case where $d = 7$ and $\gamma = (-5 + \sqrt{7}\mathrm{i})/2 = [(1 - \sqrt{7}\mathrm{i})/2]^3$. This yields

$$q(x,y) = -x^2 + \left(\frac{1 - \sqrt{7}\mathrm{i}}{2}\right)y^2 \tag{15}$$

which represents 1 in all non-dyadic fields by Lemma 5.1. Thus, in this case we choose $\mathscr{P} = ((1 - \sqrt{7}\mathrm{i})/2)O_7$ so that $N(\mathscr{P}) = 2$. Thus $\mathbb{Q}(\sqrt{7}\mathrm{i})_{\mathscr{P}} \cong \mathbb{Q}_2$ and we construct an isomorphism

$\phi:\mathbb{Q}(i\sqrt{7})_{\mathscr{P}}\to\mathbb{Q}_2$. This is obtained by taking $\tau=(1-\sqrt{7}i)/2$ as a uniformiser so that the isomorphism ϕ is induced by $\phi(\tau)=2u$ where $u\in\mathbf{Z}_2^*$. Note that

$$-1=1+\left(\frac{1-\sqrt{7}i}{2}\right)+\left(\frac{-5+\sqrt{7}i}{2}\right)=1+\tau+\tau^3,$$

so that $1+2u+2^3u^3=-1$ in $\mathbb{Q}_2$. This implies that $u\equiv 3\,(\mathrm{mod}\,4)$. Since every unit $\equiv 1\,(\mathrm{mod}\,8)$ is a square in $\mathbb{Q}_2$ [13, p. 162], the isomorphism ϕ implies that q will represent 1 in $\mathbb{Q}(\sqrt{7}i)_{\mathscr{P}}$ if and only if $-x^2+6y^2$ represents 1 in $\mathbb{Q}_2$. By a simple computation of the Hilbert symbol $(-1,6)_2$ in the 2-adics [13, p. 164] it follows that $-x^2+6y^2$ does not represent 1 in $\mathbb{Q}_2$. Thus q does not represent 1 in $\mathbb{Q}(\sqrt{7}i)$ and AG is not isomorphic to $M_2(\mathbb{Q}(\sqrt{7}i)$.

5.3. Special

The generalised triangle groups $\Gamma(p,q;r)$ with presentation

$$\langle f,g\,|\,f^p=g^q=[f,g]^r=1\rangle \tag{16}$$

and their representations in $PSL(2,\mathbb{C})$ were studied in detail in [10]. These results can be used to both include (see next section) and exclude values of γ on the tables as candidates and the relevant result from [10] is

Theorem 5.2. *For all but five exceptional triples, the groups $\Gamma(p,q;r)$ have a faithful discrete representation in $PSL(2,\mathbb{C})$ which has the parameters $(-2\cos\pi/r,-4\sin^2\pi/p,-4\sin^2\pi/q)$ for $2\leqslant p\leqslant q\leqslant\infty$, $2\leqslant r\leqslant\infty$. Furthermore $\Gamma(p,q;r)$ has a faithful finite covolume discrete representation if and only if $(p,q;r)=(3,3;3),(3,4;2),(4,4;2)$.*

Note that when $p,q,r=\infty$, the representation is such that the corresponding element is parabolic.

To eliminate the values of γ labelled *special* in the tables, we show that the corresponding groups have the same parameters as generalised triangle groups described in Theorem 5.2 with infinite covolume. Since the parameters describe the group up to conjugacy, these groups cannot have finite covolume.

Example. Let $p=q=3$ and $\gamma=-3$. Then $\mathrm{tr}[f,g]=-1$ and so $\langle f,g\rangle$ is conjugate to $\Gamma(3,4;3)$ by Theorem 5.2. Then, by the second part of Theorem 5.2, this group does not have finite covolume.

6. The groups

In this section, we identify all non-cocompact arithmetic Kleinian groups with two elliptic generators of orders p,q where $p\leqslant q$. The results are given in Table 1. The elimination processes of the preceding sections, show that for $p\geqslant 3$, there are 12 possible candidates corresponding to the 12 values of γ left to be considered in the tables.

Recall that, in all cases, $kG = \mathbb{Q}(\alpha)$ and, as at (9), we have chosen α in some $\mathbb{Q}(\sqrt{-d})$. Furthermore the relationship between α,γ emerges from equations (8) and (9) e.g for $p = q = 3$, this is given at (12). Thus in each of the 12 cases, we obtain that $kG = \mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{3}\mathrm{i})$. All traces in $G = \langle f, g\rangle$ are integer polynomials in tr$f$, tr$g$, trfg. In addition, trfg satisfies a monic polynomial with coefficients in $\mathbb{Z}[\mathrm{tr}^2 f, \mathrm{tr}^2 g, \gamma]$. Thus, since γ is an algebraic integer, it follows that all traces in G are algebraic integers. We now consider the final property of Theorem 2.1, which, as discussed in Section 5.2, reduces to showing that $AG \cong M_2(\mathbb{Q}(\sqrt{-d}))$, which is equivalent to showing that the quadratic form at (13) represents 1 in $\mathbb{Q}(\sqrt{-d})$. In all 12 cases, this is straightforward, e.g. when $p = 3, q = 4, \gamma = -2$, then $k = \mathbb{Q}(i)$. Then choosing $\beta_1 = \mathrm{tr}^2 g - 4$ gives the quadratic form $-x^2 - 3y^2 = 1$ which clearly has a solution in $\mathbb{Q}(i)$. Thus all 12 candidates give subgroups of arithmetic groups and it remains to show that each such G is of finite covolume.

In addition to the 12 above, in the cases where $p = q$, there are, up to conjugacy, one or two $\mathbb{Z}_2$-extensions of these groups which then have generators of orders 2 and p (see Section 2). Recall that, if $\langle f, g\rangle$ has $o(f) = 2, o(g) = p$, then in $\langle g, fgf\rangle$

$$\gamma(g, fgf) = \gamma(f, g)(\gamma(f, g) - \beta(g)). \tag{17}$$

Reversing this procedure we obtain the γ values in the $2, p$ cases from the corresponding γ value in the p, p case.

The groups are identified in that we give a presentation of each of the groups in this section, and in the next section, we describe the orbifold structure in most cases. The identification is obtained by comparison with groups already considered in the literature and known to have finite covolume. As noted before, for fixed p, q, the parameter γ determines the conjugacy class of the group. From the information already obtained, the groups will all be commensurable with the Bianchi groups $PSL(2, O_1)$ or $PSL(2, O_3)$. For ease of reference, rather than using the parameters described at (2), we will describe the groups equivalently by the triple (p, q, γ). In almost all cases, the description which is obtained here is that given in Table 1 in the column headed "Notation".

Note using Theorem 5.2, we can identify the groups (3,3, $-$ 3), (3,4, $-$ 2), (4,4, $-$ 2) with the generalised triangle groups $\Gamma(3,3;3)$, $\Gamma(3,4;2)$, $\Gamma(4,4;2)$ described at (16). Indeed these groups were already shown to be arithmetic in [10] and their covolumes were calculated. Since the γ value of the groups (3,3, $-$ 3), (4,4, $-$ 2) is real, the γ value of the $\mathbb{Z}_2$-extensions are complex conjugates so that, up to conjugacy in these cases, there is only one $\mathbb{Z}_2$-extension [8]. These are the groups $(2,3,(-3+\sqrt{3}\mathrm{i})/2)$ and $(2,4,-1+\mathrm{i})$, respectively. Their presentations are easily obtained from those of $\Gamma(3,3;3)$ and $\Gamma(4,4;2)$.

In a similar vein, the generalised triangle groups $\tilde{\Gamma}(p, q; r)$ with presentation

$$\langle f, g \,|\, f^p = g^q = (fgf^{-1}gfg^{-1})^r = 1\rangle \tag{18}$$

were studied in detail in [11] where the following theorem was proved.

Theorem 6.1. *If* $1/p + 1/q + 1/r = 1$, *then* $\tilde{\Gamma}(p, q; r)$ *can be embedded as a non-cocompact discrete subgroup of* $PSL(2, \mathbb{C})$ *of finite covolume.*

The proof also shows that the embedding is achieved with matrix representatives X, Y having $\mathrm{tr}\, X = 2\cos \pi/p$, $\mathrm{tr}\, Y = 2\cos \pi/q$, $\mathrm{tr}\, XYX^{-1}YXY^{-1} = -2\cos \pi/r$.

Recall the following trace identities in $SL(2,\mathbb{C})$:

$$\gamma(X,Y) = \mathrm{tr}[X,Y] - 2 = \mathrm{tr}^2 X + \mathrm{tr}^2 Y + \mathrm{tr}^2 XY - \mathrm{tr}\,X\,\mathrm{tr}\,Y\,\mathrm{tr}\,XY - 4,$$

$$\mathrm{tr}(XYX^{-1}YXY^{-1}) = (\mathrm{tr}\,X\,\mathrm{tr}\,Y - \mathrm{tr}\,XY)(-\gamma(X,Y)) + \mathrm{tr}\,XY.$$

There are six groups described by the theorem: $\tilde{\Gamma}(2,3;6)$, $\tilde{\Gamma}(2,6;3)$, $\tilde{\Gamma}(3,6;2)$, $\tilde{\Gamma}(3,3;3)$, $\tilde{\Gamma}(2,4;4)$, $\tilde{\Gamma}(4,4;2)$. For each of these six groups, we can calculate γ using these identities and from that note that each γ coincides with one on our list, so that we have identified a further six groups. For each of the groups $\tilde{\Gamma}(2,3;6)$, $\tilde{\Gamma}(2,6;3)$, $\tilde{\Gamma}(2,4;4)$ there is a subgroup generated by $\langle g,fgf\rangle$. In the first and last cases, these are the groups $\tilde{\Gamma}(3,3;3)$, $\tilde{\Gamma}(4,4;2)$ but in the second case, we note that $\langle g,fgf\rangle = \langle f,g\rangle$ so that we obtain the same arithmetic group, with two different presentations on two generators of finite order. The alternative presentation is

$$\langle f,g \mid f^6 = g^6 = (fg^{-1}fgf^{-1}gf)^2 = g^{-1}fg^{-1}fgf^{-1}gfg^{-1}fgf^{-1}g^{-1}fg^{-1}f^{-1}gf^{-1} = 1\rangle. \tag{19}$$

From these six groups we thus have three groups with presentations where $p = q$. Each of these three groups has two possible $\mathbb{Z}_2$-extensions, one of which, in each case, is already determined. Note that the γ value in these three groups where $p = q$ is not real, so that the γ values of the pair of $\mathbb{Z}_2$-extensions are not complex conjugates. Thus for example, $\tilde{\Gamma}(3,3;3) = (3,3,-2+\sqrt{3}i)$ and the $\mathbb{Z}_2$-extensions are $(2,3,(-5+\sqrt{3}i)/2) = \tilde{\Gamma}(2,3;6)$ and $(2,3,(-1+\sqrt{3}i)/2)$. In the same way we obtain the groups $(2,4,i)$ and $(2,6,(1+\sqrt{3}i)/2)$. Presentations of these groups are easily obtained by adjoining the element h where $h^2 = 1$ and $hfh = g^{-1}$, but the one coming from (19) is rather complicated (see Table 1).

Various Coxeter groups in $\mathbb{H}^3$ are well known to be commensurable with the Bianchi groups $PSL(2,O_1)$ and $PSL(2,O_3)$ [4, Appendix] and we use these to make further identifications. The group $PGL(2,O_3)$ is itself a Coxeter group with symbol $\circ \overset{3}{—} \circ \overset{3}{—} \circ \overset{6}{—} \circ$.

From the standard presentation

$$\langle x,y,z \mid x^3 = y^3 = z^6 = (xy)^2 = (xyz)^2 = (yz)^2 = 1\rangle,$$

it is not difficult to see that this group is generated by x, z and that the corresponding γ value is -1. The same applies to the groups with Coxeter symbols $\circ \overset{4}{—} \circ \overset{3}{—} \circ \overset{6}{—} \circ$ and $\circ \overset{6}{—} \circ \overset{3}{—} \circ \overset{6}{—} \circ$ which are commensurable with the group $PSL(2,O_3)$ and give the groups $(4,6,-1)$ and $(6,6,-1)$. This last group has a $\mathbb{Z}_2$-extension $(2,6,(-1+\sqrt{3})/2)$ with presentation

$$\begin{aligned}\langle f,g \mid f^2 &= g^6 = ((gf)^2(g^{-1}f)^2)^3 = ((gf)^2(g^{-1}f)^2g)^2 \\ &= (fg^{-1}f(gf)^2(g^{-1}f)^2)^2 = (fg^{-1}f(gf)^2(g^{-1}f)^2g)^2 = 1\rangle.\end{aligned} \tag{20}$$

The group $PGL(2,O_1)$ has Coxeter symbol $\circ \overset{3}{—} \circ \overset{4}{—} \circ \overset{4}{—} \circ$. With the notation as above, the subgroup $\langle x,z\rangle$ has index 10 in $PGL(2,O_1)$ and presentation

$$\langle x,z \mid x^3 = z^4 = ((xz^{-1})^2(x^{-1}z)^2)^2 = 1\rangle$$

and γ value $2i$, yielding the group Γ_{13} in Table 1.

Finally, returning to the group $\tilde{\Gamma}(2,3;6)$ with presentation as at (18) the subgroup generated by $a = g$ and $b = fgf^{-1}gfg^{-1}$ has index 4 in this group and presentation

$$\langle a,b \mid a^3 = b^6 = (ababab^{-1}a^{-1}b^{-1}a^{-1}b^{-1})^2 = 1\rangle.$$

From this we obtain that $\gamma = 2+\sqrt{3}i$ and the group Γ_{16}.

We have thus obtained all 21 presentations given in Table 1 and ascertained that they are non-cocompact arithmetic Kleinian groups. It will be noted that 15 of them are generalised triangle groups. As we have seen the last group has an alternative description as a generalised triangle group. Of the remaining five, three are Coxeter groups and the others are $\mathbb{Z}_2$-extensions of the two $p = q = 6$ groups. None of these appear to be generalised triangle groups.

It will be noted that Table 1 also gives an approximation to the covolume of these groups. These covolumes have already been computed for the generalised triangle groups $\Gamma(p,q;r)$ described in Theorem 5.2 [10]. The covolumes are also well known for the Coxeter groups (e.g. [4, Appendix]). The remaining covolumes can all be computed from the information above and the following additional calculations, made using MAGMA. (Recall that the conjugacy class of a finite covolume group is determined by its isomorphism class.)

- $\tilde{\Gamma}(2,3;6)$ is a subgroup of index 8 in $PGL(2,O_3)$.
- $\tilde{\Gamma}(2,6;3)$ is a subgroup of index 12 in $PGL(2,O_3)$.
- $\tilde{\Gamma}(3,6;2)$ is a subgroup of index 20 in $PGL(2,O_3)$.
- $\tilde{\Gamma}(2,4;4)$ is a subgroup of index 6 in $PGL(2,O_1)$.

Although we have listed 21 groups and we know that two of these groups are the same, we have not yet proved that all the others are pairwise non-conjugate. For many, this follows immediately by consideration of the covolumes. For pairs that have the same covolume, particularly pairs that are $\mathbb{Z}_2$-extensions of the same underlying group, we use the fact that the trace field of the group itself, which can be determined from the normalisation at (5), is a conjugacy invariant. By computing these trace fields, as required, we show that all other pairs are indeed non-conjugate.

7. The orbifolds

Certain orbifolds whose singular sets are graphs consisting of knots or links with tunnels were studied in [9,11,12], and the fundamental groups of these orbifolds are generalised triangle groups. The quotient spaces of the 15 generalised triangle groups admit such a description. Indeed they all admit a uniform type of description. Consider the orbifold whose singular set in S^3 is given by the graph below, where the label i indicates a cone angle of $2\pi/i$ on that edge of the singular set (see Fig. 3).

The fundamental group of this orbifold has presentation

$$\langle x, y \mid x^p = y^q = W(x,y)^r = 1 \rangle,$$

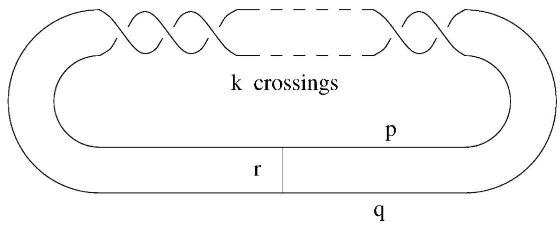

Fig. 3.

where $W(x, y)$ has $2k$ letters, k with exponent $+1$ and k with exponent -1 and

$$W(x,y) = \begin{cases} xyxy \dots xyx^{-1}y^{-1} \dots x^{-1}y^{-1} & \text{if } k \text{ is even,} \\ xyxy \dots yxy^{-1}x^{-1} \dots x^{-1}y^{-1} & \text{if } k \text{ is odd} \end{cases}$$

(cf. [12]). It is not difficult to see that all 15 of our generalised triangle groups have such presentations. When k is even, these orbifolds were discussed in [12,9]; when $k = 3$ in [11] and when $k = 5$ in [9].

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